THE BOUNDARY OF THE MILNOR FIBRE OF COMPLEX AND REAL ANALYTIC NON-ISOLATED SINGULARITIES

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Abstract. Let f and g be holomorphic functions vanishing at the origin of the affine space of dimension three. Suppose that the singular set of $(fg)^{-1}(0)$ is 1-dimensional and that the real analytic function $f\bar{g}$ has an isolated critical value at 0. By results of A. Pichon and J. Seade the function $f\bar{g}$ has a Milnor fibration. We prove that the boundary of the Milnor fibre is a Waldhausen manifold. As a intermediate milestone we describe geometrically the Milnor fibre of functions of type $f\bar{g}$ defined in the complex plane, and prove an A'Campo-type formula for the zeta function of their monodromy.

1. Introduction

It is classically known that there is a rich interplay between 3-manifold theory and the topology of isolated singularities in complex surfaces. This goes back to the work of F. Klein by the end of the 19th century, and then made clearer by in the early 1960s, by the work of Grauert [4] and Mumford [11]. A closed oriented 3-manifold M is the link of some isolated complex surface singularity (V, p) if and only if M is a Waldhausen manifold with negative definite intersection form.

This important theorem has played, on one hand, a key role for understanding the topology of surface singularities through the work of W. Neumann and many others. On the other hand, the links of isolated surface singularities provide a very interesting class of 3-manifolds which, thanks to their rich algebraic nature, have proved to be rather useful for 3-manifold theory, as for instance for the understanding of the Casson invariant, Seiberg-Witten invariants, Floer homology and other important invariants of 3-manifolds that have been discovered in the last decades.

In this sense, it is interesting to find new classes of 3-manifolds, besides the links of isolated complex surfaces singularities, that have a rich (possibly algebraic) geometric structure. And in that sense one has the interesting theorem of F. Michel, A. Pichon, A. Nemethi and A. Szilard, stating that if f is a holomorphic map-germ $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with a 1-dimensional critical set, then the boundary of the Milnor fiber is a Waldhausen manifold. The theorem was announced first by F. Michel and A. Pichon [5], but its proof contained a gap. In [8] F. Michel, A. Pichon and C. Weber provided a proof valid for some classes of singularities. The first

Date: 20-6-2011.
2000 *Mathematics Subject Classification.* Primary: 14B05, 14J17, 14E15, 32S05, 32S25, 32S45.
Research partially supported by the ERC Starting Grant project TGASS and by Spanish Contract MTM2007-67908-C02-02. The authors thank the Facultad de Ciencias Matemáticas of the Universidad Complutense de Madrid for the excellent working conditions. The second named author received financial support from CONACYT, Mexico, and CNRS, France through the Laboratorio Internacional Solomon Lefschetz and an ECOS-ANUIES agreement, as well as through grants from CONACYT and from the Posgrado en Matemáticas of Universidad Nacional Autónoma de México.

complete proof of the genaral case of the theorem was provided by A. Nemethi and A. Szilard in a long paper, where even an algorithm to compute the graph describing the Waldhausen manifold is given. Shortly afterwards F. Michel and A. Pichon have provided another complete proof which is more in the spirit of the original method they proposed.

In this article we envisage a similar problem. Consider f, g holomorphic map-germs such that the germ $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ has an isolated critical value at 0. Notice that $f\bar{g}$ has necessarily non-isolated critical points. Yet, we know from [15] that the germ $f\bar{g}$ has the Thom a_f-property, and therefore it has a Milnor fibration:

$$f : N(\epsilon, \delta^*) \to \delta^*,$$

where ϵ is a small closed ball around 0 in $\mathbb{C}^3$, δ is a disc in $\mathbb{C}$ of sufficiently small radius with respect to ϵ, and $\delta^* := \delta \backslash \{0\}$. Our main theorem (Theorem 5) in this article states that the boundary of the corresponding Milnor fibre is a Waldhausen manifold.

Although our proof has some inspiration from the method of Nemethi and Szilard, and has some points in common with that of Michel and Pichon, it provides a much shorter proof of the theorem for the holomorphic case which inmediately generalises to non-holomorphic real analytic germs of the form $f\bar{g}$. It is based in a detailed understanding of the Milnor fibre of a germ of the form $f\bar{g}$ defined in the plane in terms of an embedded resolution of $\{fg = 0\}$. Such a detailed understanding allows to generalise for real analytic germs of type $f\bar{g}$ defined in the plane, the A'Campo formula for computing the zeta-function of the monodromy in terms of an embedded resolution of singularities. This is Theorem 9.

A natural next step, which is open by now, is to find an algorithm to determine the graph associated to the boundary of the Milnor fibre.

The authors wish to thank to A. Pichon and J. Seade, who suggested the problem and held useful conversations with us, and specially to A. Pichon who detected a gap in a previous version of this paper.

2. A description of the Milnor fibre of $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ in terms of its embedded resolution

Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that $(fg)^{-1}(0)$ has an isolated singularity at the origin and the real analytic germ given by $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has an isolated critical value at $0 \in \mathbb{C}$.

In [14] Pichon and Seade prove that there exist sufficient small positive reals $0 < \delta << \epsilon$ such that the restriction

$$f\bar{g}_| : (f\bar{g})^{-1}(\mathbf{D}_\delta \backslash \{0\}) \cap \mathbf{B}^4_\epsilon \to \mathbf{D}_\delta \backslash \{0\}$$

is a locally trivial fibration, where $\mathbf{D}_\delta$ denotes the disk of radio δ in $\mathbb{C}$ centered at 0 and $\mathbf{B}^4_\epsilon$ denotes the ball in $\mathbb{C}^2$ of radius ϵ centered at 0.

Let $\pi : \tilde{M} \to B^4_\epsilon$ be a good embedded resolution of $(fg)^{-1}(0)$ and consider:

- $\pi^{-1}(0) := E = \bigcup_{i=1}^s E_i$, the exceptional divisor with its decomposition in irreducible components;
- $(f\bar{g} \circ \pi)^{-1}(0) = (fg \circ \pi)^{-1}(0) = \sum_{i=1}^s k_i E_i + \tilde{C}$ the total transform, where $\tilde{C}$ is the strict transform, which has a decomposition into connected components $\tilde{C} = \bigcup_{p=1}^w \tilde{C}_p$;
- $F := (f\bar{g} \circ \pi)^{-1}(\delta)$, the Milnor fibre of $f\bar{g}$.

For each i, $1 \leq i \leq s$, let U_i be a tubular neighbourhood of E_i, and for each p, $1 \leq p \leq w$ let $\tilde{U}_p$ be a tubular neighbourhood of $\tilde{C}_p$. Then we define the sets

- $V_{ij} := U_i \cap U_j$;
- $\tilde{V}_{ip} := U_i \cap \tilde{U}_p$;
- $V_i := U_i \backslash \big(\bigcup_{\substack{i,j=1 \\ i \neq j}}^{s} V_{ij} \cup \bigcup_{p=1}^{w} \tilde{V}_{ip} \big)$;
- $\tilde{V}_p := \tilde{U}_p \backslash \bigcup_{i=1}^{s} \tilde{V}_{ip}$;

We decompose the Milnor fibre F as follows:

$$F = \Big(\bigcup_{i=1}^{s} (V_i \cap F) \Big) \cup \Big(\bigcup_{\substack{i,j=1 \\ i \neq j}}^{s} (V_{ij} \cap F) \Big) \cup \Big(\bigcup_{p=1}^{w} (\tilde{V}_p \cap F) \Big) \cup \Big(\bigcup_{\substack{1 \leq i \leq s \\ 1 \leq p \leq w}} (\tilde{V}_{ip} \cap F) \Big)$$

Note that doing some convenient change of coordinates, each part $V_{ij} \cap F$ or $\tilde{V}_{ip} \cap F$ of the Milnor fibre F has equation of the form

$$x^{a_i} \bar{x}^{b_i} y^{a_j} \bar{y}^{b_j} \varphi_1 \overline{\varphi_2} = \delta$$

and each part $V_i \cap F$ or $\tilde{V}_p \cap F$ of the Milnor fibre has equation of the form

$$x^{a_i} \bar{x}^{b_i} \varphi_1 \overline{\varphi_2} = \delta,$$

where $\varphi_1(x,y)$ and $\varphi_2(x,y)$ are units in $\mathbb{C}\{x,y\}$, a_i is the multiplicity of f corresponding to E_i and b_i is the multiplicity of g corresponding to E_i (obviously either $a_p = 1$ and $b_p = 0$ or $a_p = 0$ and $b_p = 1$, for each $p = 1, \dots w$).

Lemma 1. *The intersection of the Milnor fibre F with each neighbourhood V_i, V_{ij}, $\tilde{V}_p$ or $\tilde{V}_{ip}$ is either a finite disjoint union of cylinders (cases (i), (iii) and (iv) in the proof) or it is a finite cover over a disk minus some disks (case (ii) in the proof).*

Proof. There are four cases to consider:

(i) $F \cap V_{ij}$ with $a_i \neq b_i$ and $a_j \neq b_j$; and $F \cap \tilde{V}_{ip}$ with $a_i \neq b_i$:
By some change of coordinates we can locally consider

$$(f\bar{g} \circ \pi) = x^{a_i} \bar{x}^{b_i} y^{a_j} \bar{y}^{b_j} \varphi_1 \overline{\varphi_2},$$

where

$$\begin{cases} \varphi_1(x,y) = \alpha + \psi_1(x,y) \\ \varphi_2(x,y) = \beta + \psi_2(x,y) \end{cases},$$

with $\alpha, \beta \in \mathbb{C}^*$ and $\psi_1(0) = \psi_2(0) = 0$. If we set

$$\begin{cases} \varphi_{1,t}(x,y) = \alpha + t\psi_1(x,y) \\ \varphi_{2,t}(x,y) = \beta + t\psi_2(x,y) \end{cases},$$

we can define the 1-parameter family

$$h_t = f_t \overline{g_t} = x^{a_i} \bar{x}^{b_i} y^{a_j} \bar{y}^{b_j} \varphi_{1,t} \overline{\varphi_{2,t}},$$

which gives a homotopy between $h_1 = f\bar{g} \circ \pi$ and $h_0 = x^{a_i} \bar{x}^{b_i} y^{a_j} \bar{y}^{b_j} \alpha \bar{\beta}$. Consider the real analytic mapping

$$\begin{array}{rccc} H: & V_{ij} \times [0,1] & \to & \mathbb{C} \times [0,1] \\ & (z,t) & \mapsto & \big(h_t(z), t\big) \end{array}$$

and its restriction

$$H_| : \big(V_{ij} \times [0,1]\big) \cap H^{-1}\big(\mathbf{D}_\delta \backslash \{0\} \times [0,1]\big) \to \mathbf{D}_\delta \backslash \{0\} \times [0,1].$$

Then $H_|$ has three properties:

- it is proper.
- It is a submersion on $\big(V_{ij}\times[0,1]\big)\cap H^{-1}\big(\mathbf{D}_\delta\backslash\{0\}\times[0,1]\big)$:
  The Jacobian matrix of H is given by

$$\begin{pmatrix} \frac{\partial h_t}{\partial x} & \frac{\partial h_t}{\partial \bar x} & \frac{\partial h_t}{\partial y} & \frac{\partial h_t}{\partial \bar y} & \frac{\partial h_t}{\partial t} \\ \frac{\partial \overline{h_t}}{\partial x} & \frac{\partial \overline{h_t}}{\partial \bar x} & \frac{\partial \overline{h_t}}{\partial y} & \frac{\partial \overline{h_t}}{\partial \bar y} & \frac{\partial \overline{h_t}}{\partial t} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial \bar x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial \bar y} & \frac{\partial t}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_t}{\partial x}\overline{g_t} & f_t\overline{\frac{\partial g_t}{\partial x}} & \frac{\partial f_t}{\partial y}\overline{g_t} & f_t\overline{\frac{\partial g_t}{\partial y}} & x^{a_i}y^{a_j}\psi_1\bar g_t \\ \bar f_t\frac{\partial g_t}{\partial x} & \overline{\frac{\partial f_t}{\partial x}}g_t & \bar f_t\frac{\partial g_t}{\partial y} & \overline{\frac{\partial f_t}{\partial y}}g_t & x^{b_i}y^{b_j}\psi_2\bar f_t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

  Then $H_|$ is a submersion in a point p if, and only if, it is not a solution of at least one of the four following equations:

$$\begin{cases} (1) \quad |\frac{\partial f_t}{\partial x}|^2|g_t|^2 - |\frac{\partial g_t}{\partial x}|^2|f_t|^2 = 0 \\ (2) \quad |\frac{\partial f_t}{\partial y}|^2|g_t|^2 - |\frac{\partial g_t}{\partial y}|^2|f_t|^2 = 0 \\ (3) \quad |f_t|^2\frac{\partial g_t}{\partial x}\overline{\frac{\partial g_t}{\partial y}} - |g_t|^2\frac{\partial f_t}{\partial x}\overline{\frac{\partial f_t}{\partial y}} = 0 \\ (4) \quad fg\big(\frac{\partial f_t}{\partial x}\frac{\partial g_t}{\partial y} - \frac{\partial f_t}{\partial y}\frac{\partial g_t}{\partial x}\big) = 0 \end{cases}$$

  Note that $f_t = x^{a_i}y^{a_j}\varphi_{1,t}$ and $g_t = x^{b_i}y^{b_j}\varphi_{2,t}$. Then setting

$$\begin{cases} \zeta_1 = a_i\varphi_{1,t} + x\frac{\partial\varphi_{1,t}}{\partial x} \\ \zeta_2 = a_j\varphi_{1,t} + y\frac{\partial\varphi_{1,t}}{\partial y} \\ \zeta_3 = b_i\varphi_{2,t} + x\frac{\partial\varphi_{2,t}}{\partial x} \\ \zeta_4 = b_j\varphi_{2,t} + y\frac{\partial\varphi_{2,t}}{\partial y} \end{cases}$$

  we have that

$$\begin{cases} \frac{\partial f_t}{\partial x} = x^{a_i-1}y^{a_j}\zeta_1 \\ \frac{\partial f_t}{\partial y} = x^{a_i}y^{a_j-1}\zeta_2 \\ \frac{\partial g_t}{\partial x} = x^{b_i-1}y^{b_j}\zeta_3 \\ \frac{\partial g_t}{\partial y} = x^{b_i}y^{b_j-1}\zeta_4 \end{cases}$$

  Substituting on equations (1) to (4), we have the equations:

$$\begin{cases} (1) \quad |x|^{2(a_i+b_i-1)}|y|^{2(a_j+b_j)}\begin{vmatrix} |\zeta_1|^2 & |\zeta_3|^2 \\ |\varphi_{1,t}|^2 & |\varphi_{2,t}|^2 \end{vmatrix} = 0 \\ (2) \quad |x|^{2(a_i+b_i)}|y|^{2(a_j+b_j-1)}\begin{vmatrix} |\zeta_2|^2 & |\zeta_4|^2 \\ |\varphi_{1,t}|^2 & |\varphi_{2,t}|^2 \end{vmatrix} = 0 \\ (3) \quad \bar x y|x|^{2(a_i+b_i-1)}|y|^{2(a_j+b_j-1)}\begin{vmatrix} \zeta_1\overline{\zeta_2} & \zeta_3\overline{\zeta_4} \\ |\varphi_{1,t}|^2 & |\varphi_{2,t}|^2 \end{vmatrix} = 0 \\ (4) \quad x^{2a_i+2b_i-1}y^{2a_j+2b_j-1}\varphi_{1,t}\varphi_{2,t}\begin{vmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{vmatrix} = 0 \end{cases}$$

  Since

$$\begin{vmatrix} |\zeta_1(0)|^2 & |\zeta_3(0)|^2 \\ |\varphi_{1,t}(0)|^2 & |\varphi_{2,t}(0)|^2 \end{vmatrix} = |a_i^2 - b_i^2|.|\varphi_{1,t}(0)\varphi_{2,t}(0)|^2$$

  and

$$\begin{vmatrix} |\zeta_2(0)|^2 & |\zeta_4(0)|^2 \\ |\varphi_{1,t}(0)|^2 & |\varphi_{2,t}(0)|^2 \end{vmatrix} = |a_j^2 - b_j^2|.|\varphi_{1,t}(0)\varphi_{2,t}(0)|^2,$$

  the result follows.

- It is a submersion on the boundary
$$\partial\big(V_{ij} \times \overline{\mathbf{D}}_{\epsilon_2} \times [0,1]\big) \cap H^{-1}\big(\mathbf{D}_\xi \backslash \{0\} \times [0,1]\big).$$
Indeed, the tubular neighbouhoods can be chosen so that V_{ij} is the polydisc $D_{\epsilon_1} \times D_{\epsilon_2}$ for small ϵ_i. Note that $\delta << min\{\epsilon_1, \epsilon_2\}$. Then $\partial(\overline{\mathbf{D}}_{\epsilon_1} \times \overline{\mathbf{D}}_{\epsilon_2}) = (\partial\overline{\mathbf{D}}_{\epsilon_1} \times \overline{\mathbf{D}}_{\epsilon_2}) \cup (\overline{\mathbf{D}}_{\epsilon_1} \times \partial\overline{\mathbf{D}}_{\epsilon_2})$ and $h_t^{-1}(\delta)$ intersects $\partial\overline{\mathbf{D}}_{\epsilon_1} \times \overline{\mathbf{D}}_{\epsilon_2}$ transversally if, and only if, the Jacobian matrix
$$\begin{pmatrix} \frac{\partial h_t}{\partial x} & \frac{\partial h_t}{\partial \bar{x}} & \frac{\partial h_t}{\partial y} & \frac{\partial h_t}{\partial \bar{y}} \\ \frac{\overline{\partial h_t}}{\partial x} & \frac{\overline{\partial h_t}}{\partial \bar{x}} & \frac{\overline{\partial h_t}}{\partial y} & \frac{\overline{\partial h_t}}{\partial \bar{y}} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial \bar{x}} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial \bar{y}} \\ \frac{\partial \bar{x}}{\partial x} & \frac{\partial \bar{x}}{\partial \bar{x}} & \frac{\partial \bar{x}}{\partial y} & \frac{\partial \bar{x}}{\partial \bar{y}} \end{pmatrix}$$
has determinant not zero in $p \neq 0$ small, which happens if equation (1) holds, and therefore if $a_i \neq b_i$. So we conclude that $h_t^{-1}(\delta)$ intersects $\partial\overline{\mathbf{D}}_{\epsilon_1} \times \overline{\mathbf{D}}_{\epsilon_2}$ transversally and, by an analogous argument, that $h_t^{-1}(\delta)$ intersects $\overline{\mathbf{D}}_{\epsilon_1} \times \partial\overline{\mathbf{D}}_{\epsilon_2}$ transversally if $a_j \neq b_j$.

Then if follows from Ehresmann's Fibration Lemma that the Milnor fibre of $h_1 = f\bar{g} \circ \pi$ is diffeomorphic to the Milnor fibre of $h_0 = x^{a_i}\bar{x}^{b_i}y^{a_j}\bar{y}^{b_j}\alpha\bar{\beta}$, which is diffeomorphic to
$$\left\{x^{a_i}\bar{x}^{b_i} = \frac{\delta}{y^{a_j}\bar{y}^{b_j}}\right\} \cap (D_{\epsilon_1} \times D_{\epsilon_2}),$$
which is a covering over an annulus of degree $|a_i - b_i|$, and therefore it is a disjoint union of at most $|a_i - b_i|$-cylinders.

(ii) $F \cap V_i$ with $a_i \neq b_i$ and $F \cap \tilde{V_p}$:
We can apply exactly the same proof of case (i), considering $(f\bar{g} \circ \pi) = x^{a_i}\bar{x}^{b_i}\varphi_1\overline{\varphi_2}$. Then we get that the Milnor fibre of $f\bar{g}$ inside V_i is diffeomerphic to the set
$$\{x^{a_i}\bar{x}^{b_i} = \delta\},$$
and therefore it is an $|a_i - b_i|$-covering of $V_i \cap E_i$, which is a disk minus some disks.

(iii) $F \cap V_{ij}$ with $a_i = b_i$ and $a_j \neq b_j$ or with $a_i \neq b_i$ and $a_j = b_j$; and $F \cap \tilde{V_{ip}}$ with $a_i = b_i$:
Consider the mapping germ
$$\begin{array}{rccc} (f \circ \pi, g \circ \pi): & (\mathbb{C}^2, 0) & \to & (\mathbb{C}^2, 0) \\ & (x, y) & \mapsto & (x^{a_i}y^{a_j}\varphi_1, x^{b_i}y^{b_j}\varphi_2) \end{array}$$
We want to find a change of coordinates $\Theta = (\Theta_1, \Theta_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that
$$(f \circ \pi, g \circ \pi) = (x^{a_i}y^{a_j}, x^{b_i}y^{b_j}) \circ \Theta,$$
which happens if and only if $(f \circ \pi, g \circ \pi) = (\Theta_1^{a_i}\Theta_2^{a_j}, \Theta_1^{b_i}\Theta_2^{b_j})$. If we set $\Theta_1 = x\theta_1$ and $\Theta_2 = y\theta_2$, with $\theta_1(0) \neq 0$ and $\theta_2(0) \neq 0$, then our problem is to find $\theta_1, \theta_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ such that
$$(f \circ \pi, g \circ \pi) = (x^{a_i}y^{a_j}\theta_1^{a_i}\theta_2^{a_j}, x^{b_i}y^{b_j}\theta_1^{b_i}\theta_2^{b_j}).$$
This happens if and only if the system
$$\begin{cases} x^{a_i}y^{a_j}\theta_1^{a_i}\theta_2^{a_j} = x^{a_i}y^{a_j}\varphi_1 \\ x^{b_i}y^{b_j}\theta_1^{b_i}\theta_2^{b_j} = x^{b_i}y^{b_j}\varphi_2 \end{cases}$$

has solution (θ_1, θ_2). This is equivalent to

$$\begin{cases} \theta_1^{a_i}\theta_2^{a_j} = \varphi_1 \\ \theta_1^{b_i}\theta_2^{b_j} = \varphi_2, \end{cases}$$

which has solution, if and only if the linear system with indeterminates $log\theta_1, log\theta_2$

$$\begin{cases} a_i \log\theta_1 + a_j \log\theta_2 = \log\varphi_1 \\ b_i \log\theta_1 + b_j \log\theta_2 = \log\varphi_2 \end{cases}$$

has solution. But we have the non-vanishing of the determinant

$$\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \neq 0,$$

and so the system has solutions. Then $(f\bar{g} \circ \pi) = (x^{a_i}\bar{x}^{b_i}|y|^{2b_i}) \circ \Theta$ and therefore its Milnor fibre is given by the equation

$$\{|y|^{2b_j} = \frac{\delta}{x^{a_i}\bar{x}^{b_i}}\} \cap V_{ij},$$

which is a disjoint union of $|a_i - b_i|$ cylinders (which intersect $\partial\overline{V}_{ij}$ transversally).

(iv) $F \cap V_i$ with $a_i = b_i$ and $F \cap V_{ij}$ with $a_i = b_i$ and $a_j = b_j$:
A. Pichon and J. Seade [14] proved that $a_i = b_i$ implies that E_i does not represent a rupture vertex of the dual graph of the total transform of $(fg)^{-1}(0)$ by π.
Let S be the union of all the exceptional divisors E_i such that $a_i = b_i$ and consider its decomposition in connected components $S = S_1 \cup \cdots \cup S_k$. For each S_l, let Ω_l be the union of all the V_i, V_{ij} and $\tilde{V_{ip}}$ that intersect S_l, excluding the V_{ij}'s that intersect some E_i with $a_i \neq b_i$ (there are at least one and at most two of them). There are only two cases to consider:
(a) There is just one V_{ij} in S_l that intersects some E_i with $a_i \neq b_i$.
(b) There are exactly two V_{ij}'s in S_l that intersect some E_i's with $a_i \neq b_i$.
We claim that case (a) does never occur. In fact, we know that if M denotes the $(s \times s)$-intersection matrix of E, that is,

$$m_{ij} = \begin{cases} E_i^2, \text{ if } i = j; \\ 1, \text{ if } i \neq j \text{ and } E_i \text{ intersects } E_j \\ 0, \text{ otherwise} \end{cases},$$

and if u_i denotes the number of intersection points between E_i and the strict transform of f minus the number of intersection points between E_i and the strict transform of g, then (see [3], Theorem 18.2)

$$M \cdot \begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_s - b_s \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_s \end{pmatrix} = 0$$

So in case (a) we would have

$$\begin{pmatrix} E_1^2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & E_2^2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E_{s-1}^2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & E_s^2 \end{pmatrix} \cdot \begin{pmatrix} a_1 - b_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_1 \\ 0 \\ \vdots \\ 0 \\ u_s \end{pmatrix} = 0,$$

which implies that $a_1 = b_1$, a contradiction. In fact, it follows that there cannot be two consecutive vertices with $a_i = b_i$, that is, for each l one has $S_l = E_i$ and $\Omega_l = V_i$.

It follows that the piece

$$(f\bar{g} \circ \pi_{|\Omega_l})^{-1}(\partial \mathbf{D}_\delta) \cap \Omega_l$$

of the Milnor tube $(f\bar{g} \circ \pi_{|\Omega_l})^{-1}(\partial \mathbf{D}_\delta)$ which is contained in Ω_l is diffeomorphic to a solid torus minus a small tubular neighbourhood of its core, and therefore it has the homotopy of a torus $\mathbb{T}^2$. The restriction

$$(f\bar{g} \circ \pi)_| : (f\bar{g} \circ \pi)^{-1}(\partial \mathbf{D}_\delta) \cap \Omega_l \to \partial \mathbf{D}_\delta$$

is the projection of a locally trivial fibre bundle. Then supposing that $F \cap \Omega_l$ is connected, we get

$$\pi_2(\mathbf{S}^1) \to \pi_1(F \cap \Omega_l) \to \pi_1(\mathbb{T}^2) \to \pi_1(\mathbf{S}^1) \to \pi_0(F \cap \Omega_l),$$

which is isomorphic to

$$0 \to \pi_1(F \cap \Omega_l) \to \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \to 0.$$

This gives that $\pi_1(F \cap \Omega_l)$ is isomorphic to $\mathbb{Z}$, and then $F \cap \Omega_l$ is a cylinder. If the piece of the Milnor fibre $F \cap \Omega_l$ has more than one connected component, since the Milnor tube $(f\bar{g} \circ \pi)^{-1}(\partial \mathbf{D}_\delta \setminus \{0\}) \cap \Omega_l$ is connected, the monodromy m must define a one-cycle permutation $(F_1, \ldots, F_r)$ on the connected components of $F \cap \Omega_l$. In particular, they are all diffeomorphic. For $i \neq r$, let m_i denote the diffeomorphism from F_i to F_{i+1}, and let m_r be the diffeomorphism from F_r to F_1. Then we can construct a fibre bundle over the circle with total space

$$(f\bar{g} \circ \pi)^{-1}(\partial \mathbf{D}_\delta) \cap \Omega_l$$

and with fibre F_1 in the following way: consider the circle given by the identification of the extrema of the interval $[0, r]$. The space

$$(f\bar{g} \circ \pi)^{-1}(\partial \mathbf{D}_\delta) \cap \Omega_l$$

is diffeomorphic to the quotient of $\coprod_{i=0}^{r-1} [i, i+1] \times F_{i-1}$ by the identification of (i, x) with $(i+1, m_i(x))$ for any $i < r$ and (r, x) with $(0, m_r(x))$. The projection to the circle is now obvious.

∎

We shall need the following Remark on families of functions of the form $f_t\bar{g}_t$, which is an adaptation of the corresponding result for families of holomorphic functions.

Remark 2. *Let $f_t : (\mathbb{C}^2, 0) \to \mathbb{C}$, $g_t : (\mathbb{C}^2, 0) \to \mathbb{C}$ be families of holomorphic germs depending holomorphically on a parameter t which varies in a disc D. Suppose that $f_t g_t$ has an isolated singularity at the origin whose Milnor number is independent*

of t. For any $t_0 \in D$ there exists a neighbourhood $t_0 \subset U \subset D$, and a positive number ϵ which is a Milnor radius for $f_t\bar{g}_t$, for any $t \in U$.

Proof. A Milnor radius for a real analytic germ of the form $f\bar{g}$ is a positive radius ϵ such that for any other radius $\epsilon' \leq \epsilon$, the sphere of radius ϵ' centered at the origin meets $fg^{-1}(0)$ transversely. Thus ϵ is a Milnor radius for the real analytic germ $f\bar{g}$ if and only if it is a Milnor radius for the holomorphic germ fg. The assertion of the Remark follows from the corresponding one for μ-constant families of holomorphic germs of functions in two variables: its well known that a μ-constant family of plane curves is Whitney equisingular and admits a uniform Milnor radius. ∎

3. The boundary of the Milnor fibre of non-isolated singularities

Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that the real analytic map-germ $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ has an isolated critical value at $0 \in \mathbb{C}$. Seade and Pichon [15] prove that such real analytic function $f\bar{g}$ as above always have Thom's a_F-property. Hence there exist $\epsilon, \delta \in \mathbb{R}$ sufficiently small, $0 < \delta << \epsilon$, such that $f\bar{g}$ can be given a Milnor-Lê fibration in the tube as follows:

$$f\bar{g}_{|} : (f\bar{g})^{-1}(\mathbf{D}_\delta \backslash \{0\}) \cap \mathbf{B}^6_\epsilon \to \mathbf{D}_\delta \backslash \{0\}.$$

We would like to prove that the boundary of the Milnor fibre

$$L_t := (f\bar{g})^{-1}(t) \cap \mathbf{S}^5_\epsilon,$$

for $t \in \mathbf{D}_\delta \backslash \{0\}$ is a Waldhausen manifold.

3.1. **Systems of neighbourhoods.** In the holomorphic case it is well known that it is possible to define the Milnor fibration using different systems of neighbourhoods at the origin. Balls and polydisks are the most widely used systems of neighbourhoods. In the proof of our main result we need to work with a Milnor fibration defined using a polydisk instead of a ball. We shall prove now that the boundary of the Milnor fibre defined for polydisks is homeomorphic to the boundary of the Milnor fibre defined for balls.

Let Σ be the singular set of $\{fg = 0\}$. It is the complex curve given by

$$\Sigma(f\bar{g}) = \Sigma(f) \cup \Sigma(g) \cup \big(f^{-1}(0) \cap g^{-1}(0)\big) = \Sigma(fg).$$

Choose a coordinate system (x, y, z) of $\mathbb{C}^3$ such that there exists ϵ such that for any $\epsilon' \leq \epsilon$ the boundary of the polydisk

$$\Delta_\epsilon = \{(x, y, z) \in \mathbb{C}^3 : \max\{|x|, |y|, |z|\} \leq \epsilon\}$$

meets Σ transversely at the open face

$$\{(x, y, z) \in \mathbb{C}^3 : \max\{|x|, |y|\} < \epsilon, |z| = \epsilon\}.$$

Consider the family of norms in $\mathbb{C}^3$

$$||(x, y, z)||_s := (|x|^{1/s} + |y|^{1/s} + |z|^{1/s})^s,$$
$$||(x, y, z)||_0 := \max\{|x|, |y|, |z|\}$$

for $s \in [0, 1/2]$, which depends continuosly on the parameter s.

A positive number ϵ is a Milnor radius for the function $\{fg = 0\}$ with respect to the norm $||\,\centerdot\,||_s$ if for any positive $\epsilon' \leq \epsilon$ the hypersurface $fg = 0$ is transverse in the stratified sense to the sphere $||(x, y, z)||_s = \epsilon'$. It is well known that for any $s \in [0, 1/2]$ there is a Milnor radius for $\{fg = 0\}$ with respect to the norm $||\,\centerdot\,||_s$. Moreover, by the continuity of the norms in the parameter s, if ϵ is a Milnor radius

for $\{fg = 0\}$ with respect to the norm $||\centerdot||_s$, then there exists a neighbourhood U of $s \in [0, 1/2]$ such that ϵ is a Milnor radius for $\{fg = 0\}$ with respect to the norm $||\centerdot||_{s'}$ for any $s' \in U$. Using the compactness of $[0, 1/2]$ we find a radius ϵ which is a Milnor radius for $\{fg = 0\}$ with respect to the norm $||\centerdot||_s$ for any $s \in [0, 1/2]$.

Given any $U \subset [0, 1/2]$ we define the set

$$\mathcal{B}^U_\epsilon := \{(x, y, z, s) \in \mathbb{C}^3 \times U : ||(x, y, z)||_s \leq \epsilon\}.$$

It must be viewed as a family of Milnor balls for varying norms.

Since the function $f\bar{g}$ satisfies the Thom a_F-property, for any $s \in [0, 1/2]$ there exists a neighbourhood U of $s \in [0, 1/2]$ and a positive δ such that the mapping

$$F^U : \mathcal{B}^U_\epsilon \cap (f\bar{g})^{-1}(D_\delta \setminus \{0\}) \to D_\delta \setminus \{0\} \times U$$

defined by $F^U(x, y, z, s) := (f(x, y, z)\bar{g}(x, y, z), s)$ is a topological locally trivial fibration.

By compactness of $[0, 1/2]$ there exists a positive δ such that the mapping

$$F : \mathcal{B}^{[0,1/2]}_\epsilon \cap (f\bar{g})^{-1}(D_\delta \setminus \{0\}) \to D_\delta \setminus \{0\} \times [0, 1/2]$$

is a topological locally trivial fibration.

Consequently, the boundaries of fibres $F^{-1}(t, 0)$ and $F^{-1}(t, 1/2)$ are homeomorphic for any $t \in D_\delta \setminus \{0\}$, and hence the boundary of the Milnor fibre defined for polydiscs is homeomorphic to the boundary of the Milnor fibre defined for balls.

3.2. **The Waldhausen structure.** Our main goal is to prove that the boundary of the Milnor fibre

$$L_t = (f\bar{g})^{-1}(t) \cap \mathbf{S}^5_\epsilon,$$

for $t \in \mathbf{D}_\delta \backslash \{0\}$ is a Waldhausen manifold. Since we have proved that the boundary of the Milnor fibre defined using balls is homeomorphic to the boundary of the Milnor fibre using polydisk, from this moment on we will assume that $\mathbf{S}^5_\epsilon$ denotes the boundary of the ball of Milnor radius ϵ for the norm $||\centerdot||_0$ as in 3.1.

The singular set of

$$L_0 = (f\bar{g})^{-1}(0) \cap \mathbf{S}^5_\epsilon = (fg)^{-1}(0) \cap \mathbf{S}^5_\epsilon$$

is the intersection of the sphere $\mathbf{S}^5_\epsilon$ with the complex curve Σ. We know that $L(\Sigma) := \Sigma \cap \mathbf{S}^5_\epsilon$ is a finite disjoint union of circles $\mathbf{S}^1$ contained in the open face

$$\{(x, y, z) \in \mathbb{C}^3 : \max\{|x|, |y|\} < \epsilon, |z| = \epsilon\}.$$

Let

$$n : \tilde{X} \to (fg)^{-1}(0)$$

be the normalization of $(fg)^{-1}(0)$ and set $\tilde{\Sigma} := n^{-1}(\Sigma)$.

Let W denote the vanishing zone of $f\bar{g}$, which is nothing but a tubular neighbourhood of $L(\Sigma)$ in $\mathbf{S}^5_\epsilon$, and we define $W_0 := W \cap L_0$ and $W_t := W \cap L_t$. Then it is well known that $\tilde{X}\backslash\tilde{W}_0$ is a Waldhausen manifold, where $\tilde{W}_0 := n^{-1}(W_0)$ is a tubular neighbourhood of $\tilde{\Sigma}$ in $\tilde{X}$ (see [5] for instance). Since the normalization is an isomorphism outside W_0, we find that $L_0\backslash W_0$ is a Waldhausen manifold.

It is also easy to check that $L_t\backslash W_t$ is diffeomorphic to $L_0\backslash W_0$, and then it follows that $L_t\backslash W_t$ is Waldhausen. Since ∂W_t is a finite disjoint union of tori, all we have to prove is that W_t is a Waldhausen manifold.

Now, if $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$ is the decomposition of Σ into irreducible components, we get the decomposition into disjoint connected components $W = W[1] \sqcup \cdots \sqcup W[k]$,

where $W[i]$ is a small tubular neighbourhood of the circle $\Sigma_i \cap \mathbf{S}^5_\epsilon$ in $\mathbf{S}^5_\epsilon$, for $i = 1, \dots, k$.

Fix a component Σ_l. Given $p \in \Sigma_l \setminus \{0\}$ let H_p be the 2-dimensional affine hyperplane of $\mathbb{C}^3$ passing through p and parallel to $\{z = 0\}$. Choosing ϵ small enough we may assume that the Milnor number of the germ $(fg|_{H_p}, p)$ is independent of p. Therefore, by Remark 2 and the compactness of $\Sigma_l \cap \mathbf{S}^5$ we deduce the existence of a positive η such that for any $p \in \Sigma_l \cap \mathbf{S}^5$ the ball in H_p centered at p and of radius η is a Milnor ball for $(f\bar{g})|_{H_p}$ at p. We may choose $W[l]$ to be the union of those balls when p varies in $\Sigma_l \cap \mathbf{S}^5$. With this definition there is an obvious fibration

$$\sigma_l : W[l] \to \Sigma_l \cap \mathbf{S}^5$$

with fibre a complex 2-ball.

Since $f\bar{g}$ satisfies Thom's a_F-property there exists a positive δ such that the mapping

$$\Psi_l : W[l] \cap (f\bar{g})^{-1}(D_\delta) \to D_\delta \times (\Sigma_l \cap \mathbf{S}^5)$$

defined by $\Psi_l := (f\bar{g}, \sigma_l)$ has only the circle $\{0\} \times (\Sigma_l \cap \mathbf{S}^5)$ as critical values. Therefore, for $t \in D_\delta \setminus \{0\}$, the restriction

$$\sigma_l|_{L_t \cap W[l]} : L_t \cap W[l] \to \Sigma_l \cap \mathbf{S}^5$$

is a locally trivial fibration. Its fibre is called the Transversal Milnor fibre of $f\bar{g}$ at Σ_l and its monodromy h the vertical monodromy along Σ_l, see [16].

To prove that W_t is a Waldhausen manifold, we have to show that each connected component $L_t \cap W[l]$, for $i = 1, \dots, k$, is a Waldhausen manifold. To do that, it is sufficient to give a decomposition of each transversal Milnor fibre which is invariant under the corresponding vertical monodromy such that the corresponding pieces of $L_t \cap W[l]$ are Seifert manifolds. We will prove that they are Seifert manifolds either by proving that they are fibrations with base a circle and fibre a cylinder, or by showing directly that the restriction of h to the corresponding piece of transversal Milnor fibre is periodic.

Let us fix on an irreducible component Σ_l of Σ, which by an easy argument can be assumed, without loosing generality, to be the z-axis (see [7], Lemma 4.4). Let D be the disk of radius ϵ around the origin of Σ_l. This coincides with the intersection of Σ_l with the polydisc of size ϵ. Let $\mathbf{S}^1$ be its boundary circle. The region $W[l]$ coincides now with the product $\mathbf{S}^1 \times B_\eta$, where B_η is the ball of radius η in the (x, y)-complex 2-plane, and the mapping σ_l coincides with the projection to the first factor.

We can look at the restriction $(f\bar{g})_| : D^* \times \mathbb{C}^2 \to \mathbb{C}$ as a family in the parameter D^*. We denote by $f\bar{g}_s$ the restriction $f\bar{g}_{|\{s\}\times\mathbb{C}^2}$. The corresponding holomorphic family $fg_| : D^* \times \mathbb{C}^2 \to \mathbb{C}$ is μ-constant over D^*. Then we can consider a minimal embedded resolution in family

$$\pi : \tilde{M} \to D^* \times \mathbb{C}^2$$

where

- $\pi^{-1}(D^* \times \{0\}) := E$ is the exceptional divisor, with a decomposition in irreducible components $E = \cup_{i=1}^r E_i$, where an irreducible component is defined as the closure of a connected component of $E \setminus Sing(E)$;
- for each $s \in \mathbf{S}^1$ define $X_s := \pi^{-1}(\{s\} \times B_\eta)$. Then

$$\pi_s : X_s \to \mathbb{C}^2$$

is the minimal embedded resolution of the plane curve singularity that the restriction of fg to H_s defines at the point $(s,0,0)$. We denote by E^s the excepcional divisor of π_s, and by E_i^s the set of irreducible components of E^s contained in E_i.

For each $i \in \{1,\dots,r\}$, let U_i be a tubular neighbourhood of E_i and define the boxes

$$V_{ij} := U_i \cap U_j \quad \textit{and} \quad V_i := U_i \backslash \cup_{i\neq j} V_{ij}.$$

Then for each $s \in \mathbf{S}^1$, the transversal Milnor fibre of $(f\bar{g})^{-1}(t) \cap \sigma_l^{-1}(s)$ is diffeomorphic to $(f\bar{g}_s \circ \pi_s)^{-1}(t)$, and this set can be decomposed as follows:

$$(f\bar{g}_s \circ \pi_s)^{-1}(t) = \left(\bigcup_i \left(V_i \cap ((f\bar{g}_s \circ \pi_s)^{-1}(t)\right)\right) \cup \left(\bigcup_{i,j} \left(V_{ij} \cap (f\bar{g}_s \circ \pi_s)^{-1}(t)\right)\right).$$

This decomposition is preserved by the vertical monodromy.

In the holomorphic case (when g is constant), we know that each part of the Milnor fibre of type $V_{ij} \cap (f_s \circ \pi_s)^{-1}(t)$ has equation of the form $x^{k_i}y^{k_j} = t$, and therefore it is a finite union of cylinders. In Lemma 1 we show that the same happens in the general case, that is, $V_{ij} \cap (f\bar{g}_s \circ \pi_s)^{-1}(t)$ is also a finite disjoint union of cylinders. Hence $V_{ij} \cap (f\bar{g})^{-1}(t) \cap \pi^{-1}(\mathbf{S}^1 \times \mathbb{C}^2)$ is a fibre bundle over $\mathbf{S}^1$ with fibre a finite disjoint union of cylinders. The classification of such kind of fibrations yields that all their total spaces are Seifert manifolds.

In the general case, the V_i's satisfying $a_i = b_i$ are grouped in a finite number of connected regions Ω_k and by Lemma 1, case (iv), we have that $\Omega_k \cap (f\bar{g}_s \circ \pi_s)^{-1}(t)$ is a disjoint union of cylinders. As before we deduce that the corresponding piece is a Seifert manifold.

In the holomorphic case, each part of the Milnor fibre of type $V_i \cap (f_s \circ \pi_s)^{-1}(t)$ has equation of the form $x^{k_i} = t$, and therefore $V_i \cap (f \circ \pi_s)^{-1}(t)$ is a finite covering over $E_i^s \cap V_i$.

We showed in Lemma 1, case (ii), that when $a_i \neq b_i$ the very same happens in the general case, that is,

$$\begin{array}{c} V_i \cap (f\bar{g}_s \circ \pi_s)^{-1}(t) \\ \Big\downarrow{\scriptstyle \rho_i} \\ E_i^s \cap V_i = \mathbb{P}^1 \backslash \mathrm{d}isks \end{array}$$

is a finite covering, where ρ_i is the projection of the tubular neighbourhood of E_i^s to E_i^s.

If the connected components of E_i^s do not represent rupture vertices of the dual graph of the total transform of the transversal singularity by π_s, then $V_i \cap (f\bar{g})_s^{-1}(t)$ is a finite disjoint union of either disks or cylinders, and therefore $V_i \cap (f\bar{g})^{-1}(t) \cap \pi^{-1}(\mathbf{S}^1 \times \mathbb{C}^2)$ is a fibre bundle over $\mathbf{S}^1$ with fibre a finite disjoint union of either disks or cylinders, and therefore it is a Seifert manifold. So now we shall see what happens in V_i if the connected components of E_i^s represent rupture vertices.

Proposition 3. (*i*) *A finite cover of a Seifert manifold is a Seifert manifold;* (*ii*) *A finite cover of a Waldhausen manifold is a Waldhausen manifold.*

Proof. Let $\pi : M' \to M$ be a finite cover of a Waldhausen manifold. Write $M = \cup M_i$, where each Seifert piece M_i is a fibre bundle over a compact surface with boundary F_i and fibre $\mathbf{S}^1$ (with finitely many multiple special fibres), and projection

$p_i : M_i \to F_i$. It is enough to prove that each piece $M_i' := \pi^{-1}(M_i)$ is a Seifert manifold. The composition

$$p_i \circ \pi : M_i' \to F_i$$

is a bundle (with finitely many multiple special fibres) with fibre a finite cover of $\mathbf{S}^1$. If such fibre is connected, then we have expressed M_i' as a bundle over a surface with fibre $\mathbf{S}^1$, and therefore it is a Seifert manifold.

If the fibre is not connected, we define the following equivalence relation in M_i': two points are identified if they belong to the same connected component of the same fibre of $p_i \circ \pi$. The quotient of M_i' by this equivalence relation is a surface F_i' that covers F_i, and the quotient application $q : M_i' \to F_i'$ expresses M_i' as a bundle over F_i' with fibre $\mathbf{S}^1$.

■

Now fix $i \in \{1, \ldots, r\}$ and note that the component E_i fibres over the punctured disk D^* with projection

$$\pi_1 : E_i \to D^*,$$

where π_1 is the composition of the mapping π restricted to E_i and the projection $p(x, y, z) = z$ of $\mathbb{C}^3$ to the z-axis Σ_l. Its fibre E_i^s is a disjoint union of $\mathbb{P}^1$'s. According to Proposition 3, all we have to proof is that $E_i \cap V_i \cap \pi_1^{-1}(\mathbf{S}^1)$ is Waldhausen.

To do that, we would like π_1 to have connected fibre. It is easy to produce a finite cover

$$\tau : D^* \to D^*$$

and an analytic lift

$$\pi_2 : E_i \to D^*$$

which is a locally trivial bundle with fibre $\mathbb{P}^1$ such that $\tau \circ \pi_2 = \pi_1$.

Lemma 4. *If E_i represents a rupture vertex of the dual graph of π, then, after pullback by a finite covering of the base, the lift $\pi_2 : E_i \to D^*$ is the projection of a trivial bundle.*

Proof. Let $\varsigma_1, \varsigma_2, \varsigma_3 : D^* \to E_i$ be three pairwise disjoint sections of π_2 given by the intersection of E_i with three distinct components of the total transform of $(fg)^{-1}(0)$ by π (either other exceptional divisors or components of the strict transform). Note that it may be necessary to make a base change of D^* so the sections are uni-valued.

Consider the fibre bundle homeomorphism $\Psi : E_i \to \mathbb{P}^1 \times D^*$ defined by sending three pairwise different constant sections of the first bundle to the sections $\varsigma_1, \varsigma_2, \varsigma_3$ on the second and imposing that fibrewise Ψ is a projective isomorphism.

■

From the previous Lemma it is inmediate that the restriction of the vertical monodromy $h_| : E_i^s \to E_i^s$, for $s \in \mathbf{S}^1$, is periodic (it is the identity after a finite base change). Therefore $E_i \cap V_i \cap \pi_1^{-1}(\mathbf{S}^1)$ is a Seifert manifold. We have proved:

Theorem 5. *Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that the real analytic germ given by $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ has an isolated critical value at $0 \in \mathbb{C}$. Then the boundary of the Milnor fibre of $f\bar{g}$ is a Waldhausen manifold.*

4. The Zeta Function of the Monodromy

Now we want to give a formula to calculate zeta function of the monodromy h of $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with an isolated singularity, in terms of the combinatorics

of the embedded resolution of fg, in the same way A'Campo [1] did to calculate the monodromy of holomorphic functions.

If we set $F_\theta := (f\bar{g})^{-1}(\delta e^{i\theta})$, for δ sufficiently small, then for each $q \geq 0$, the monodromy $h : F_\theta \to F_\theta$ defines an isomorphism between vector spaces (the homology groups) given by $h^* : H^q(F_\theta;\mathbb{C}) \to H^q(F_\theta;\mathbb{C})$, the so called *algebraic monodromy.*

The *zeta function* of such monodromy is defined by

$$Z(t) = \prod_{q\geq 0} \left(det(Id^* - th^*)\right)^{(-1)^{q+1}}.$$

There is a very classic way of calculating the zeta function of h in terms of the Lefschetz numbers of h, which we describe below:

For each $k \geq 1$, the Lefschetz number of the k-iteration of h is defined by

$$\Lambda(h^k) = \sum_{q\geq 0} (-1)^q trace[(h^*)^k : H^q(F_\theta, \mathbb{C}) \to H^q(F_\theta, \mathbb{C})].$$

If we define the integers $s_1, s_2, \dots$ by the relations

$$\Lambda(h^k) = \sum_{i|k} s_i,$$

then the zeta function of h is given by

$$Z(t) = \prod_{i\geq 1} (1 - t^i)^{-s_i/i}.$$

So all we have to do is to calculate the Lefschetz numbers of h. First we recall the following lemma:

Lemma 6. *Consider the following commutative chain map on an exact sequence:*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & G_0 & \xrightarrow{\alpha_0} & G_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-1}} & G_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle \varphi_n} & & \\
0 & \longrightarrow & G_0 & \xrightarrow{\alpha_0} & G_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-1}} & G_n & \longrightarrow & 0
\end{array}$$

Then

$$\sum_{i=0}^{n} (-1)^i \mathrm{trace}[\varphi_i] = 0$$

Let $\pi : \tilde{M} \to \mathbb{C}^2$ be an embedded resolution of the germ $(fg)^{-1}(0)$ at the origin. Let $E = \cup_{i=1}^{s}$ be a decomposition of the exceptional divisor of π in irreducible components. Let a_i and b_i denote the multiplicity of E_i in the total transform of $f^{-1}(0)$ and $g^{-1}(0)$ respectively.

We apply the previous lemma to the Mayer-Vietoris sequence associated to the decomposition of the Milnor fibre of $f\bar{g}$ in the boxes V_i and V_{ij} and Ω_l as in Lemma 1 (here, in order to simplify notation, we do not make distinction between V_i and $\tilde{V_p}$ nor between V_{ij} and $\tilde{V_{ip}}$) we get:

$$\Lambda(h^k) = \sum_{\substack{i=1\\ a_i\neq b_i}}^{s} \Lambda(h^k_{V_i\cap F_\theta}) + \sum_{l} \Lambda(h^k_{\Omega_l\cap F_\theta}) + \sum_{\substack{i,j=1\\ i\neq j}}^{s} \Lambda(h^k_{V_{ij}\cap F_\theta}) - \sum_{i=1}^{s} \Lambda(h^k_{\partial V_i\cap F_\theta}).$$

Recall that we have grouped the components V_i with $a_i = b_j$ and V_{ij} with $a_i = b_i$ and $a_j = b_j$ in larger domains Ω_l, and we have proved in Lemma 1 that the part

of the Milnor number contained in these components is a finite union of cylinders. Moreover, we have seen that each of these domains Ω_l contains only one V_i, that is, $\Omega_l = V_i$, with $a_i = b_i$.

Lemma 7. *We have:*

(*i*) $\Lambda(h^k_{V_{ij}\cap F_\theta}) = 0$, *for any* $i \neq j$;
(*ii*) $\Lambda(h^k_{\Omega_l\cap F_\theta}) = 0$, *for any* l;
(*iii*) $\Lambda(h^k_{\partial V_i\cap F_\theta}) = 0$, *for any* i.

Proof. Each of the pieces of the Milnor fibre considered are either finite unions of cylinders (cases (i) and (ii)) or of circles (case (iii)), by the way the Milnor fibre have been splitted (see Lemma 1). The cylinders of cases (i) and (ii) have always a circle of case (iii) as a boundary component. Since a cylinder can be retracted to its boundary, it is enough to prove the result for case (iii). The finite union of circles $\partial V_i \cap F_\theta$ is a finite covering over a circle of E_i and the monodromy is compatible with the covering projection. Since the Euler characteristic of the circle vanishes, the result then follows from the following lemma: □

Lemma 8. *Let* $\pi : X \to B$ *be a* m*-covering of a compact manifold with boundary and let* h *be an automorphism of* X *such that the diagram*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\pi"'] & X \arrow[dl, "\pi"] \\ B & \end{tikzcd}$$

commutes. For $b \in B$*, denote by* $h_b : \pi^{-1}(b) \to \pi^{-1}(b)$ *the permutation induced by* h*, and suppose this permutation is cyclic and transitive. Then*

$$\Lambda(h^k) = \chi(B).\Lambda(h_b^k) = \begin{cases} \chi(B).m, & if\ m \mid k; \\ 0, & if\ m \nmid k \end{cases}$$

Proof. Suppose that B is contractible. Then $X \simeq B \times \pi^{-1}(b)$ and then $H^q(X) \simeq H^q(\pi^{-1}(b))$ and therefore $\Lambda(h^k) = \Lambda(h_b^k)$. If B is not contractible, we can write it as a (finite) union of contractible sets B_i, $i \in \{1, \dots, c\}$, such that $B_i \cap B_j$ is contractible, for any $i, j \in \{1, \dots, c\}$. Then we proceed by induction on c.

If the result is true for $c-1$, we define $\hat{B} = \cup_{i=1}^{c-1} B_i$, $X_{cup} = \pi^{-1}(\hat{B})$ and $X_c = \pi^{-1}(B_c)$. Then write $B = \hat{B} \cup B_c$ and applying the Mayer-Vietoris sequence associated to this decomposition one gets

$$\Lambda(h) = \Lambda(h_{|X_{cup}}) + \Lambda(h_{|X_c}) - \Lambda(h_{|X_{cup}\cap X_c}) =$$
$$\chi(\hat{B})\Lambda(h_b^k) + \chi(B_c)\Lambda(h_b^k) - \chi(\hat{B}\cap B_c)\Lambda(h_b^k) =$$
$$\chi(B)\Lambda(h_b^k).$$

Now observe that $\Lambda(h_b^k) = \sum_{q\geq 0}(-1)^q trace[(h_b^k)^* : H^q(\pi^{-1}(b)) \to H^q(\pi^{-1}(b))]$, which is the trace of the induced isomorphism $(h_b^k)^* : \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{m-times} \to \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{m-times}$, which is equal to m if $m \mid k$, or zero otherwise. ■

Now, if $a_i \neq b_i$, then $V_i \cap F_\theta$ is a degree $d_i = |a_i - b_i|$-covering over E_i minus r_i-disks, where r_i is the number of double points of the total transorm of $(fg)^{-1}(0)$

on E_i. Moreover, the monodromy h is compatible with the covering projection. Hence using the two previous lemmata we get the following formula:

$$\Lambda(h^k) = \sum_{\substack{i=1 \\ a_i \neq b_i}}^{s} \Lambda(h^k_{X_i}) = \sum_{\substack{i=1 \\ a_i \neq b_i \\ d_i | k}}^{s} d_i(2 - r_i).$$

Moreover, since h^0 denotes the identity, we also have

$$\Lambda(h^0) = \sum_{i=1}^{s} \Lambda(h^0_{V_i \cap F_\theta}) = \sum_{i=1}^{s} \chi(V_i \cap F_\theta) = \sum_{i=1}^{s} d_i(2 - r_i).$$

Now, since for each $k \geq 1$ we have

$$\sum_{d_i | k} d_i(2 - r_i) = \Lambda(h^k) = \sum_{d_i | k} s_{d_i},$$

it follows that

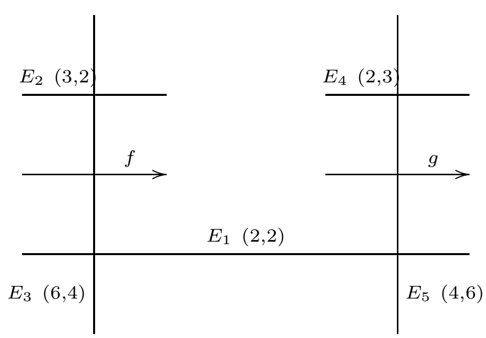
$$s_{d_i} = d_i(2 - r_i)$$

and then we have the following theorem:

Theorem 9. *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that the real analytic map-germ $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has an isolated singularity. Let $\pi : \tilde{M} \to \mathbb{C}^2$ be an embedded resolution of the germ $(fg)^{-1}(0)$ at the origin. Let $E = \cup_{i=1}^{s} E_i$ be a decomposition of the exceptional divisor of π in irreducible components. Let a_i and b_i denote the multiplicity of E_i in the total transform of $f^{-1}(0)$ and $g^{-1}(0)$ respectively. Define $d_i := |a_i - b_i|$. Let r_i be the number of double points of the total transform of fg in E_i. The the zeta function of the monodromy of the Milnor fibration of $f\bar{g}$ is given by*

$$Z(t) = \prod_{i=1}^{s} (1 - t^{d_i})^{r_i - 2}.$$

Example 10. Consider the holomorphic functions $f(x, y) = x^2 + y^3$ and $g(x, y) = x^3 + y^2$. Then the real analytic germ $f\bar{g} : (\mathbb{R}^4, 0) \to (\mathbb{R}^2, 0)$ has an isolated singularity at $0 \in \mathbb{C}^2$. The graph of the resolution of the complex curve $\{fg = 0\} = \{f\bar{g} = 0\}$ is given below:



In the holomorphic case fg, the part of the Milnor fibre inside each box V_{ij} is a disjoint union of $gcd(a_i + b_i, a_j + b_j)$ cylinders, and inside each box V_i it is an $(a_i + b_i)$-covering of a sphere minus r_i disks, with Euler characteristic $(a_i + b_i)(2 - r_i)$.

Then the part of the Milnor fibre inside: V_{13} and V_{15} are two cylinders; V_{23} and V_{45} are five cylinders; V_1 is two cylinders; V_2 and V_4 are five disks; V_3 and V_5 are compact surfaces of genus 2 with boundaries eight circles. So the Milnor fibre of fg is a twice-perforated surface of genus 5.

In the real analytic case $f\bar{g}$, according to Lemma 1, the part of the Milnor fibre inside: V_{13} and V_{15} are two cylinders; V_{23} and V_{45} are one cylinder; V_1 are two cylinders; V_2 and V_4 are one disk; V_3 and V_5 are compact surfaces of genus 0, that is, spheres, with boundaries four circles each one. Hence the Milnor fibre of $f\bar{g}$ is a twice-perforated surface of genus 1, that is, a torus with boundary two disjoint circles.

Moreover, by Theorem 9 the zeta function of h^{fg} is given by

$$Z(t) = (1 - t^5)^{-2}(1 - t^{10})^2$$

and the zeta function of $h^{f\bar{g}}$ is given by

$$Z(t) = (1 - t)^{-2}(1 - t^2)^2.$$

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